1. (4%) Let $X \sim N(0,1)$ and $E \sim N(0,1)$ be independent, and let $Y = X + \beta E$. Show the correlation coefficient of X and Y denoted by $\rho_{XY} = \frac{1}{\sqrt{\beta^2+1}}$.

2. (6%) Let T be an exponential random variable with parameter λ, and conditional on T, let U be uniform on $[0, T]$, that is, $U|T \sim Uniform(0, T)$. Find the unconditional mean and variance of U.

3. (5%) Let X_1, X_2, X_3 denote a random sample from a distribution of the continuous type having pdf

$$f(x) = \begin{cases} 2x & 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the moment generating function of X, $M(t)$, and verify that $E(X) = M'(0)$.

4. (10%) Suppose that X is a discrete random variable with $P(X = 1) = \theta$ and $P(X = 2) = 1 - \theta$. Three independent observation s of X are made: $x_1 = 1$, $x_2 = 2$, $x_3 = 2$.

   (a) (4%) Find the maximum likelihood estimator of θ.

   (b) (3%) If ● has a prior distribution that is uniform on [0, 1], find the posterior distribution.

   (c) (3%) Take the loss function to be $\mathcal{L}[\theta, \delta(x)] = (\theta - \delta(x))^2$, find the Bayes' solution $\delta(x)$ for a point estimate θ.

5. (10%) Let

$$f(x,y) = \begin{cases} (2/\theta^2)e^{-(x+y)/\theta} & 0 < x < y < \infty, \\ 0 & \text{elsewhere.} \end{cases}$$

be the joint pdf of the random variables X and Y.

(a) (4%) Find the mean and variance of Y.

(b) (6%) Given $E(Y|x) = x + \theta$, find $E(X + \theta)$ and $Var(X + \theta)$.

6. (20%) Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from a distribution with pdf

$$f(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) (5%) Find the probability that the range($= Y_4 - Y_1$) is less than $\frac{1}{2}$.

(b) (4%) Let $Z = (Y_1 + Y_4)/2$. Find the pdf of Z.

(c) (3%) Consider $W = -\theta \log X$, where $\theta > 0$. Find the pdf of W.

(d) (5%) Suppose the random sample $W_1, \ldots, W_n$ from the distribution of W. Find the the maximum likelihood estimator (MLE) of $P(W \geq 2)$).

(e) (3%) Possibly, in a life-testing situation, however, we only observe the first r order statistics $W_{(1)} < W_{(2)} < \cdots < W_{(r)}$. Write down the joint pdf of these order statistics and denote id by $L(\theta)$.

7. (12%) Let $X_1,\ X_2, \cdots,\ X_n$ be a random sample from $N(0,\ \theta)$, $0 < \theta < \infty$.

   (a) (4%) Find the MVUE of θ.

   (b) (5%) Now consider $H_0 : \theta = \theta'$, where θ' is a fixed positive number, and $H_a : \theta < \theta'$. Find an uniformly most powerful critical region for testing H_0 against H_a.

   (c) (3%) Based on the results in (b), now take $n = 15$, $\alpha = 0.05$, and $\theta' = 3$ find c=? ($\chi^2_{(14,0.05)} = 13.339, \chi^2_{(14,0.95)} = 23.685$, $\chi^2_{(15,0.05)} = 14.339, \chi^2_{(15,0.95)} = 24.996$).

8. (10%) Let $X_1,\ X_2, \cdots,\ X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \le x \le \theta, \\ 0 & \text{elsewhere} \end{cases}$$

   (a) (5%) Find the UMVUE of θ.

   (b) (2%) Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample from a uniform distribution on $(0, \theta)$, where $\theta > 0$. Show that Λ for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ is $\Lambda = (Y_n/\theta_0)^n$, $Y_n \le \theta_0$, and $\Lambda = 0$ if $Y_n > \theta_0$.

   (c) (3%) Based on the result in (b), when H_0 is true, show that $-2log\Lambda$ has an exact $\chi^2(2)$ distribution, not $\chi^2(1)$. Note that the regularity conditions are not satisfied.

9. (9%) For the simple linear regression model, $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $i = 1, \ldots, n$ and ϵ_i are i.i.d $N(0, \sigma^2)$.

   (a) (6%) To derive the MLE (Maximum Likelihood Estimator) for β_0, β_1 and σ^2.

   (b) (3%) Find an unbiased estimator of σ^2. Show the details.

10. (14%) Let $X_1,\ X_2, \cdots,\ X_n$ denote a random sample of size n from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, and $\theta > 0$.

    (a) (3%) Find the Rao-Cramer's lower bound of MLE of θ, where $\hat{\theta}_{MLE} = -n/\sum_{i=1}^{n} logX_i$.

    (b) (2%) Find an approximate (1-α)100% confidence interval for θ.

    (c) (4%) Let $W = -\sum_{i=1}^{n} logX_i$. Based on the distribution of W, obtain an exact (1-α)100% confidence interval for θ.

    (d) (5%) Find the UMUVE of θ.