For the following problems , $V^{\perp}$ denotes the orthogonal compliment of vector space V, A^T denotes the transpose of the matrix A and $p'(x)$ means the derivative of $p(x)$ with respect to x.

1. Let $V = \{[x_1, x_2] | x_1$ is a real number and x_2 is a positive number $\}$ be a vector space with vector addition defined by $[x_1, x_2] \oplus [y_1, y_2] = [x_1 + y_1, x_2 y_2]$ and scalar multiplication defined by $r[x_1, x_2] = [rx_1, (x_2)^r]$ for any $r \in R$. Find the additive identity $\vec{0}$ and the additive inverse of $\vec{v} = [-3, 2]$. (b) Show the scalar multiplication satisfies the distributive property $r(\vec{x} \oplus \vec{y}) = r\vec{x} \oplus r\vec{y}$. (10%)

2. Find a basis for the nullspace of the linear transformation $T(x_1, x_2, x_3, x_4) = [x_1 - 3x_2 + x_4, x_1 + x_2 + x_3 - x_4]^T$. (10%)

3. Let W be a subspace of R^n. Show that $R^n = W \bigoplus W^{\perp}$. (10%)

4. Let $A \in R^{4\times 4}$ with the spectrum $\sigma(A) = \{1, -1, 2, 2\}$ (collection of all eigenvalues of A). Evaluate $\det(A)$, $\text{trace}(A)$, $\det(A^2 + I)$ and $\text{trace}(A^2 + I)$ (10%)

5. Show that $\|\vec{u} + \vec{v}\|_2 \le \|\vec{u}\|_2 + \|\vec{v}\|_2$ for any $\vec{u}, \vec{v} \in R^n$. (10%)

6. Consider the vector space P_2 of polynomials of degree at most 2, and let $T : P_2 \to P_2$ be the linear transformation defined as $T(p(x)) = (x-1)^2 p''(x) + xp'(x) + p(x)$.
   (a) Find a matrix representation for T associated with the ordered basis $\{1, (x-1), (x-1)^2\}$. (5%)
   (b) Find $p(x)$ such that $T(p(x)) = 2x + 1$. (5%)
   (c) Find eigenvalues λ and the associated eigenfunctions $p(x)$ for T. (i.e. $T(p(x)) = \lambda p(x)$) (10%)

7. Let $\vec{u}$ and $\vec{v}$ be column vectors in R^n. Define the matrix $A = I - \vec{u}\vec{v}^T$. Show that $A^{-1} = I + \beta\vec{u}\vec{v}^T$ for some scalar β if $\vec{v}^T\vec{u} \ne 1$. Solve the linear system $A\vec{x} = \vec{b}$ where $\vec{u} = [1, 0, -1, 1, 0]^T, \vec{v} = [1, 0, 2, 0, -1]^T$ and $\vec{b} = [1, 2, 3, 4, 5]^T$. (15%)

8. Determine whether the statement is true or false. If it is true, prove it, otherwise, give a counter example. (15%)
   (a) Let T be a linear transformation from R^n to R^n with inner product denoted by $< \cdot, \cdot >$. If $< T(\vec{x}), \vec{x} >= 0$ for each $\vec{x} \in R^n$, then $T = 0$.
   (b) If U and V are subspaces of a vector space X, then the union of U and V is also a subspace of X.
   (c) Let $A \in R^{n\times m}$ and $B \in R^{m\times k}$, then $\text{rank}\,(AB) \le \text{rank}\,(A)$.