1. (12%) Suppose that $T:\mathbb{R}^4\to\mathbb{R}^4$ is a linear transformation such that $T(\mathbf{e}_i)=\mathbf{v}_i$ for $i=1,2,3,4$, where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ is the standard basis for $\mathbb{R}^4$ and

$$\mathbf{v}_1=(1,0,2,1),\ \mathbf{v}_2=(2,1,5,1),\ \mathbf{v}_3=(1,-1,1,2),\ \mathbf{v}_4=(1,2,4,-1).$$

Find the dimension of the range of T and find a basis for the kernel of T.

2. (18%)
   (a) Assume that the cubic curve $y=ax^3+bx^2+cx+d$ passes through the points $(-1,2)$, $(0,5)$, $(1,8)$, and $(2,23)$. Find the coefficients a, b, c, and d.
   (b) Prove that there is a unique polynomial, $y=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}$, whose graph passes through n given points (x_1,y_1), (x_2,y_2), $\ldots$, (x_n,y_n), provided that the x-coordinates are distinct.

3. (20%) Find the characteristic polynomial, minimal polynomial, and Jordan canonical form of the following matrix

$$A=\begin{bmatrix}2&1&-2&-2\\0&0&0&0\\2&2&-2&-2\\-2&-3&2&2\end{bmatrix}.$$

4. (20%) Suppose that $\mathbf{u}=(1,2,3,4,5,6)$ and $\mathbf{v}=(1,-1,1,-1,1,-1)$ are (column) vectors in $\mathbb{R}^6$ and $A=\mathbf{u}\mathbf{v}^T$, a 6×6 matrix.
   (a) Show that $\mathbf{u}$ is an eigenvector of A. What is the corresponding eigenvalue?
   (b) Show that 0 is an eigenvalue of A of (algebraic) multiplicity 5.
   (c) Compute the trace of A^6.

5. (15%) A bilinear form is defined on $V=\mathbb{R}^3$ by

$$\langle\mathbf{u},\mathbf{v}\rangle=[u_1,u_2,u_3]\begin{bmatrix}1&2&0\\2&2&2\\0&2&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

for any vectors $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ in V.
   (a) Find a basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ for V such that $\langle\mathbf{v}_i,\mathbf{v}_j\rangle=0$ for $i\neq j$.
   (b) Is it true that $\langle\mathbf{v},\mathbf{v}\rangle\geq 0$ for any $\mathbf{v}$ in V? Give reasons.

6. (15%) A square matrix A is *Hermitian* if $A^*=A$ and it is *normal* if $A^*A=AA^*$, where A^* is the conjugate transpose of A. Show that a square matrix is Hermitian if and only if it is normal and its eigenvalues are real.