(20%) 1. Let $K \subseteq R^n$. Show that the following statements are equivalent:

(1) For any open covering S of K, S has finite intersection property,

(2) K is closed and bounded in R^n,

(3) For any sequence $\{c_j\}_{j=1}^{\infty} \subseteq K$, there exists a subsequence $\{c_{j_m}\}_{n=1}^{\infty}$ so that it is convergent and the limit is in K.

(15%) 2. Show that if a real-valued function f is continuous on $[a, b]$, then

(1) f is uniformly continuous on $[a, b]$ and

(2) the range of $f = \{f(x) | x \in [a, b]\}$ is a closed interval of $(-\infty, \infty)$.

(15%) 3. Suppose a real-valued function g is continuous on $[0, \infty)$, then

(1) Give the definition that g in integrable on $[0, \infty)$, and show the following property:

(2) Suppose a real-valued function f is continuous on $[0, \infty)$ and $|f|$ is integrable on $[0, \infty)$. Define the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(x) \cos(x\xi) dx, \quad \xi \in [0, \infty].$$

Then f is integrable on $[0, \infty)$, and $\hat{f}$ is well-defined and continuous on $[0, \infty)$.

(15%) 4. Let $I = [a_1, b_1] \times [a_2, b_2]$ and a real-valued function f is defined on I. Suppose $\partial_x f$ and $\partial_y f$ is continuous on I. Also, we denote the interior of I by I^0. Show that

(1) f is continuous on I, and

(2) the θ-directional derivative $\partial_\theta f$ satisfies

$$\partial_\theta f(x_0, y_0) = \partial_x f(x_0, y_0) \cos\theta + \partial_y f(x_0, y_0) \sin\theta,$$

for $0 \le \theta < \pi$ and $(x_0, y_0) \in I^0$.

(20%) 5. (1) Applying the method of integration by parts to show the Taylor's expansion formula given as follows:
If $f, f', \ldots, f^{(n+1)}$ are real-valued and continuous on $[a, b]$, then for $c \in [a, b]$, we have

$$f(x) = f(c) + f'(c)(x - c) + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_{n+1}(x), \quad x \in [a, b],$$

and

$$R_{n+1}(x) = \frac{1}{n!} \int_c^x (x - t)^n f^{(n+1)}(t) dt, \quad x \in [a, b], \quad n \ge 0.$$

(2) Show that the exponential function satisfies

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for } x \in (-\infty, \infty).$$

(15%) 6. Let $g(x) = \sum_{n=1}^{\infty} n^2 x^n$, show that

(1) $\lim_{n\to\infty} (n^2)^{\frac{1}{n}} = 1$,

(2) The function g is continuous in $(-1, 1)$, and g can not be defined in the set $\{x \mid |x| \geq 1\}$, and

(3) g is infinitely differentiable in $(-1, 1)$.