There are 7 problems with 100 points. Show all your work for partial credits.

1. (10 pts.) Apply Gaussian elimination and back-substitution to solve the following linear system, and give the geometric description of the solution.

$$\begin{cases} x+y+z=2 \\ x+2y+z=3 \\ 2x+3y+2z=5 \end{cases}.$$

2. (10 pts.) Prove that $rank(AB) \le rank(A)$ and $rank(AB) \le rank(B)$.

3. (10 pts.) Let $A = [3 \quad 1 \quad -1]$, and let V be the nullspace of A.
   (1) Find orthonormal bases for V and $V^{\perp}$ respectively.
   (2) Find the projection matrix P_1 that projects vectors in R^3 onto $V^{\perp}$. And, find the projection matrix P_2 that projects vectors in R^3 onto V.

4. (10 pts.) Let V be the vector space of all continuous real-valued functions defined on the closed interval $[0,1]$.
   (1) Verify that $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ for $f, g \in V$ defines an inner product on V.
   (2) Prove that $T: V \to V$ defined by $T(f(x)) = \int_0^x f(t)dt$ for $0 \le x \le 1$ is a linear transformation. Is T one-to-one? Why?

5. (25 pts.)
   (1) Find the eigenvalues of matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.
   (2) Find all the eigenvectors of A. Are there 4 independent eigenvectors? Are there 4 orthonormal eigenvectors?
   (3) Find the rank and determinant of $A+2I$?
   (4) Create a nonsymmetric matrix (if possible) with eigenvalues 1, 2 and 4. Can you create a rank one matrix with those eigenvalues? Explain your answer.
   (5) Create a symmetric matrix (not diagonal) with eigenvalues 1, 2 and 4.

6. (15 pts.) Let $A_n = \begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ 1 & a_2 & -1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & 1 & a_n \end{bmatrix}$.

   (1) Show that $\det A_n = a_n \det A_{n-1} + \det A_{n-2}$ for $n \geq 3$.

   (2) Evaluate $\det A_6$ for the case $a_j = j$, $j = 1,2,\ldots,6$.

   (3) Evaluate $\det A_6$ for the case $a_j = 6 - j$, $j = 1,2,\ldots,6$.

7. (20 pts.) State TRUE or FALSE for each of the following statements. Give a reason (if true) or an counterexample (if false).

   (1) A symmetric matrix with a positive determinant is positive definite.

   (2) If A is a symmetric invertible matrix, then A^{-1} is symmetric.

   (3) For any $A, b, x,$ and y, if $Ax = 0$ and $A^T y = b$, then $x^T b = 0$

   (4) A can not be similar to $-A$ unless $A = 0$.