(15%) 1. This problem is concerned with the estimation of the variance of normal distribution with unknown mean from a sample $X_1, \dots, X_n$ of i.i.d. normal random variables. In answering the following questions, use the fact that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

and that the mean and variance of a chi-square random variable with r df are r and $2r$, respectively.

(a) Which of the following estimates is unbiased?

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

(b) Which of the estimates given in part (a) has the smaller MSE?

(c) For what value of ρ does $\rho\sum_{i=1}^{n}(X_i - \bar{X})^2$ have the minimal MSE? [Hint: $\text{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2$]

(12%) 2. The probability of a family chosen at random having exactly k children is $\alpha p^k, 0 < p < 1$. Suppose that the probability that any child has blue eyes is $b, 0 < b < 1$, independently of others. Let us write

$$p_k = \alpha p^k, k = 1, 2, \dots,$$
$$p_0 = 1 - \frac{\alpha p}{(1-p)}.$$

(a) Find the conditional probability that a family has $r(r \geq 0)$ children with blue eyes, given that it has k children.

(b) What is the probability that a family chosen at random has exactly $r(r \geq 0)$ children with blue eyes?

(c) Find the conditional probability that a family has at least two children with blue eyes, given that it has at least one child with blue eyes.

(20%) 3. Consider an i.i.d. sample of random variables with density function

$$f(x \mid \sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right).$$

(a) Find the method of moments estimate of σ.

(b) Find the maximum likelihood estimate of σ.

(c) Find the asymptotic variance of the mle.

(d) Find a sufficient statistic for σ.

(15%) 4. Let $X_1, X_2, \ldots, X_n$ be iid rv's, $X_i \sim N(\mu, \sigma^2)$. Suppose that both μ and σ^2 are unknown.

(a) Find a $100(1-\alpha)\%$ confidence interval for μ.

(b) Find a $100(1-\alpha)\%$ confindence interval for σ^2.

(c) Find a $100(1-\alpha)\%$ confidence set for (μ, σ^2).
[Hint: use Bonferroni's inequality $P(A \bigcap B) \geq 1 - P(A^c) - P(B^c)$]

(10%) 5. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$. To test the null hypothesis $H_0; \mu = \mu_0$, the t test is often used:

$$t = \frac{\bar{X} - \mu_0}{s_{\bar{X}}}$$

Under H_o, t follows a t distribution with $n-1$ df. Show that the likelihood ratio test of this H_0 is equivalent to the t test.

(15%) 6. Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(a) Find a test that has significance level $\alpha = 0$. What is its power?

(b) For $0 < \alpha < 1$, consider the test that rejects when $X \in [0, \alpha]$. What is its significance level and power?

(c) What is the significance level and power of the test that rejects when $X \in [1-\alpha, 1]$?

(d) Find another test that has the same significance level and power as the previous one.

(e) Does the likelihood ratio test determine a unique rejection region?

(13%) 7. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of iid rv's $X_1, X_2, \ldots, X_n$ with common pdf

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the distribution of $X_{(r)}$.

(b) Find the joint distribution of $X_{(j)}$ and $X_{(k)}$.

(c) Find the conditional distribution of $X_{(j)} \mid X_{(k)} = x$, and show that $X_{(j)}/(1-x)$ is beta $(k-j, n-k+1)$.