

5. (12分) Find the matrix representation of the linear transformation

$$T: \mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 - 2x_3, 2x_2 + 3x_4).$$

Find the kernel, range, nullity, and the rank of the transformation.

6. (12分) Prove that $S = \{(x, y, z) \mid 2x - 4y + 6z = 0\}$ is a subspace of \mathbb{R}^3 . Find an orthonormal basis for S .

7. (12分) Let $\mathbf{v} = [x, y]^T$. Find a symmetric matrix A such that the quadratic equation

$$3x^2 + 2xy + 3y^2 = 4 \tag{1}$$

can be written in the form $\mathbf{v}^T A \mathbf{v} = 4$. Find the principal axes of the graph of the quadratic equation (1) and identify the conic section whose equation is given in (1).

8. (12分) Let $A \in \mathbb{R}^{n \times n}$. Prove that the following statements are equivalent:

- (a) $\text{rank}(A) = n$.
- (b) A can be written as the product of elementary matrices.
- (c) The null space of A is $\{0\}$.

(請提供詳細計算或證明過程，僅有答案而沒有過程得零分!)

1. (10分) In P_3 determine whether the polynomials $\{2x^2 - x, -x^2 + 4x + 7, 8x^2 - 7x - 6\}$ are linearly dependent or independent.

2. (15分) Determine whether the given matrix is invertible. If it is, calculate the inverse.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -2 \end{pmatrix}.$$

3. (15分) Calculate eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 5 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Find an invertible matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix containing eigenvalues of A . How to compute A^{100} using this decomposition?

4. (12分) Find all solutions (if any) to the linear system:

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 - x_2 + 3x_3 = 5 \\ -x_1 + 5x_2 - 3x_3 = -4. \end{cases}$$