There are five problems with 100 points.
Show your work for partial credits.

1. (24 pts.) Let P_3 be the set of the polynomials with real coefficients of degree less than or equal to 3. Let $v_1=(1,\ 1,\ 1)^T$, $v_2=(4,\ 2,\ 1)^T$, $v_3=(3,\ 2,\ 1)^T$, we define a linear transformation $T:\Re^3 \to P_3$ by $T(v_1)=5x^3+7x^2-3x-1$, $T(v_2)=-3x^3+2x-2$, and $T(v_3)=-x^3+7x^2+x-5$.
   (a) Show that $\beta=\{v_1,v_2,v_3\}$ and $\wp=\{1,\ 1+x,\ 1+x+x^2,\ 1+x+x^2+x^3\}$ are bases for $\Re^3$ and P_3 respectively.
   (b) Find the matrix of T relative to the bases β and $\wp$.
   (c) Find a basis for the range of T. What is the dimension of the range of T?
   (d) Find a basis for the kernel of T. What is the dimension of the kernel of T?
2. (21 pts.) Let V be the subspace of $\Re^4$ that is the intersection of equations $x_1-x_2+x_4=0$ and $2x_1-x_2+x_3+x_4=0$.
   (a) Find orthogonal bases for V and $V^\perp$ respectively.
   (b) Find the matrix that projects vectors of $\Re^4$ orthogonally onto V.
   (c) Find the vector in V that is closest to $(1,\ 2,\ 1,\ 1)^T$.
3. (10 pts) Let $A=\begin{bmatrix}3 & 3 & -4\\ -4 & -3 & 5\\ 2 & 4 & 0\end{bmatrix}$.

   Evaluate $A^6+4A^5-5A^4-50A^3-76A^2-10A+50I_3$.
4. (15 pts.) Let $A=\begin{bmatrix}0.4 & 0 & 0.2\\ 0.3 & 0.8 & 0.3\\ 0.3 & 0.2 & 0.5\end{bmatrix}$. Construct the solution of the dynamical system $x_{k+1}=Ax_k$, $k=0,\ 1,\ 2,\ \cdots\cdots$, that satisfies $x_0=(0,\ 0.4,\ 0.6)$. What happen to x_k as $k\to\infty$.
5. (30 pts) Prove or disprove the following statements.
   (a) If A is an $m\times n$ matrix, B is an $n\times p$ matrix, and C is a $r\times m$ matrix, then rank CA $\geq$ rank A $\geq$ rank AB.
   (b) If a matrix A has a left inverse, the system Ax = b has a unique solution.
   (c) Let w be a vector in $\Re^n$ of length 1, then $H=I-2ww^T$ is a projection matrix.
   (d) If A is a real symmetric matrix, there exists a matrix B such that $A=BB^T$.
   (e) Let $T:\Re^n\to\Re^n$ be a linear transformation, T is onto if and only if T is invertible.