* There are 5 problems with 100 points in this test.
** Show your work for partial credits.

1. Let A be an $n \times n$ matrix. Suppose $Ax = 5x + 3y$, $Ay = 3x + 5y$, $Az = 7z + w$, and $Aw = z + 7w$ for $x, y, z, w \in R^n$. Let $B = [x \quad y \quad z \quad w]$ be the $n \times 4$ matrix.
   (a) (4 pts.) Find a 4×4 matrix C such that AB=BC.
   (b) (10 pts) Diagonalize the matrix C.
   (c) (6 pts.) Find a matrix E satisfying $E^2 = C$
   (d) (4 pts.) If B is invertible, find the determinant of A. Is A diagonalizable?

2. Let P_2 be the set of the polynomials with real coefficients of degree less than or equal to 2. Define a transformation $T : P_2 \to P_2$ as $T(a + bx + cx^2) = (2a + b + c) + (2a + b - 2c)x - (a + 2c)x^2$ for any $a, b, c \in R$.
   (a) (6 pts.) Is T a linear transformation? Is T an isomorphism?
   (b) (6 pts) Find the matrix of T relative the basis $B = \{1 - x^2, 1 + x, 2x + x^2\}$.
   (c) (6 pts.) Find the eigenspaces for T.

3. Let matrix A be row equivalent to $\begin{pmatrix} 1 & 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
   (a) (6 pts.) Find a basis for $Nul\,A$ and a basis for $(Nul\ A)^{\perp}$.
   (b) (6 pts.) If the first, second, and fourth columns of A are $(2 \quad 1 \quad 1 \quad 2)^T$, $(3 \quad 1 \quad 1 \quad 2)^T$, $(4 \quad 1 \quad 2 \quad 3)^T$, determine the matrix A.
   (c) (6 pts.) Is the vector $b = (17 \quad 6 \quad 8 \quad 14)^T$ in $Col\,A$?

4. (10 pts.) Find the least squares approximating function of the form $f(x) = ax + b2^x$ for the given data pairs (-1,0), (0,1), and (1,4).

5. (30 pts.) State <u>true</u> or <u>false</u> for each statement, and prove or disprove it.
   (a) Suppose U and V are subspaces of R^n, then $(U \cap V)^{\perp} = U^{\perp} + V^{\perp}$.
   (b) A matrix is invertible if and only if it is unitary..
   (c) Let A be an $n \times n$ matrix. If $A^2 = 0$, then $rankA \le \frac{n}{2}$.
   (d) If $V = \sum_{i=1}^{k} W_i$ and $W_i \cap W_j = \{0\}$ for $i \ne j$, then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.
   (e) $\langle A, B \rangle = tr(B^* A)$ defines an inner product for the vector space $M_{2\times2}(R)$.