1. (15 pts) Let $f \in C(\mathbb{R})$ and $f(x) = f(x+1)$ for all $x \in \mathbb{R}$.
   (a) Show that f takes on maxima and minima.
   (b) Show that there exists a x_0 such that $f(x_0 + 1/2) = f(x_0)$.

2. (15 points) Let $f \in C^1(\mathbb{R})$ and $f'(x) \geq c > 0$ for all $x \in \mathbb{R}$. Show that for each $y \in \mathbb{R}$, there is a unique x such that $f(x) = y$.

3. (15 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$f^2(t) = 2\int_0^t f(s)ds$$

for $t \geq 0$. Show that either $f \equiv 0$ or there is a $t_0 \geq 0$ such that

$$f(t) = \begin{cases} t - t_0, & t \geq t_0, \\ 0, & 0 \leq t \leq t_0. \end{cases}$$

4. (15 points) Let $G \subset \mathbb{R}^n$ be an open set and $f \in C^2(G)$. Suppose that $a \in G$ is a critical point for f. If the Hessian $D^2 f(a)$ is positive definite, show that f has a local minimum at $x = a$.

5. (15 points)

$$\text{Let} \qquad f_n(x) = x^n.$$

   (a) Show that the sequence f_n converges to zero uniformly on $[0, 1-\delta]$ for any $\delta > 0$.
   (b) Prove or disprove that the sequence f_n converges uniformly on $[0, 1]$.

6. (10 points) Let the function

$$f(x,y) = |x|^p |y|^q / \sqrt{x^2 + y^2}, \quad (x,y) \neq (0,0)$$

and set $f(0,0) = (0,0)$. For what combinations of $p \geq 0$ and $q \geq 0$ is f continuous at $(0,0)$?

7. (15 points) Let $D = \{0 \leq x, y \leq 1\}$. Show that there exists a unique solution in D to the following equations

$$\begin{cases} \sin(\frac{x+y}{2}) = x \\ 1 - \frac{x^2+y^2}{4} = y. \end{cases}$$