Algebra and Linear Algebra

Notation: Let $M_{m\times n}(K)$ denote the set of matrices of size $m \times n$ over the field K and I_n the identity matrix of size $n \times n$. Let S_n be the symmetric group of degree n, and A_n the alternating group of degree n. Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{F}_n$ the finite field with n elements.

There are six problems and totally 100 points.

1. Let $A \in M_{5\times 5}(\mathbb{C})$. Suppose that the minimal polynomial of A is $f(x) = x^4 - x^3 - x^2 + x$.
   (a) Is A diagonizable? (You need to explain your answer.) (7 points)
   (b) What are the possible Jordan forms of A? (8 points)
   (c) What is the minimal polynomial of A^2? (8 points)

2. Let $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \in M_{3\times 4}(\mathbb{C})$.
   (a) Does there exist $X \in M_{4\times 3}(\mathbb{C})$ such that $BX = I_3$? (If so, please find one; if not, please explain why not.) (8 points)
   (b) Does there exist $Y \in M_{4\times 3}(\mathbb{C})$ such that $YB = I_4$? (If so, please find one; if not, please explain why not.) (8 points)

3. Let G be a group and H, K two subgroups of G. Suppose that $|G| = 225$, $|H| = 75$, and $|K| = 45$. Show that $H \cap K$ is a normal subgroup of K. (15 points)

4. Consider $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ as a finite commutative ring with n elements, where $n \in \mathbb{N}$.
   (a) Suppose $\mathbb{Z}_n$ has only one maximal ideal. Show that $n = p^k$, where p is a prime integer and $k \in \mathbb{N}$. (7 points)
   (b) Let $R = \mathbb{Z}_{60}$ and $Z(R) = \{\, x \in R \mid \exists\, y \neq 0, \text{such that } xy = 0 \,\}$. How many elements are there in $Z(R)$? (8 points)

5. (a) Show that there exists $\alpha \in \mathbb{F}_{243}$ such that $\alpha^5 - \alpha + 1 = 0$. (8 points)
   (b) Show that $\forall\, \beta \in \mathbb{F}_{243}$, $\beta^4 - \beta^3 - \beta - 1 \neq 0$. (8 points)

6. (a) Let $\sigma = (12345), \tau = (21345) \in S_5$. Find $\gamma \in S_5$ such that $\sigma = \gamma\tau\gamma^{-1}$. (7 points)
   (b) Let $cl(\sigma) = \{\, x\sigma x^{-1} \mid x \in A_5 \,\}$. How many elements are there in $cl(\sigma)$? (8 points)