ADVANCED CALCULUS

INSTRUCTION. This exam contains 5 problems with total 100 points. Each problem is worth 20 points. To earn partial credits, show your work in detail.

1. Consider the sequence $\{a_n\}$, where $a_1 = 1$ and
$$a_{n+1} = 1 + \frac{1}{1+a_n},$$
for all $n \in N$.
   (a) (5pts) Is $\{a_n\}$ monotone?
   (b) (10pts) Use the contraction principle to show that $\{a_n\}$ converges.
   (c) (5pts) Verify that $\{a_n\}$ converges to $\sqrt{2}$.

2. Show that
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$
by obtaining a series expansion for $\tan^{-1} x$ and using Abel's theorem.

3. Is the function $\sqrt{x}$ uniformly continuous on $[0, \infty)$? Justify your claim.

4. Let $\mathbf{f} = (f_1, f_2, f_3)$ be the vector-valued function defined in $\mathbb{R}^3$ for which $x_1 + x_2 + x_3 \neq -1$ as follows:
$$f_k(x_1, x_2, x_3) = \frac{x_k}{1 + x_1 + x_2 + x_3} \quad (k = 1, 2, 3).$$
   (a) (8pts) Find $\mathbf{J}_\mathbf{f}(x_1, x_2, x_3)$.
   (b) (12pts) Show that $\mathbf{f}$ is one-to-one and compute $\mathbf{f}^{-1}$ explicitly.

5. Does the sequence of functions
$$\begin{aligned} f_n(x) : [0, \infty) &\to \mathbb{R} \\ x &\mapsto x^n e^{-nx} \end{aligned}$$
converge uniformly? Justify your claim.