1. (10%) Find a quadratic approximation for $f(x, y) = \sin(x)\sin(y)$ at (0,0) by Taylor's formula. What's the maximum error when $|x| \le 0.1$ and $|y| \le 0.1$?

2. (15%) Evaluate the integral

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 \, dy \, dx.$$

by using the Jacobian of a transformation. You may need to define $u = x + y$ and $v = y - 2x$.

3. (15%) Show that the Mean-Value Theorem can be written as follows

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h),$$

where $0 < \theta < 1$. Determine θ as a function of x and h when (a) $f(x) = x^2$, and (b) $f(x) = \log(x)$.

4. (15%) Let $f(x, \ldots, x_n) = x_1^2 \times \cdots \times x_n^2$, which represents the product of all x_i^2. Find the maximum value of $f(x_1, \ldots, x_n)$ under the condition $x_1^2 + \ldots + x_n^2 = 1$. Use this result to show the following inequality

$$(a_1 \cdots a_n)^{1/n} \le \frac{a_1 + \cdots + a_n}{n}.$$

5. (18%) Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find an orthonormal basis for the column space of A.

(b) Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

(c) Solve the least squared problem $A\boldsymbol{x} = \boldsymbol{b}$.

6. (12%) Show that a symmetric matrix A is positive definite if and only if all of its eigenvalues are positive.

7. (15%) We define a linear transformation L from R^3 into R^3 by

$$L(c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3) = (c_1 + c_2 + c_3)\boldsymbol{y}_1 + (2c_1 + c_3)\boldsymbol{y}_2 - (2c_2 + c_3)\boldsymbol{y}_3,$$

where

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It is obvious to see that $S = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is a basis for R^3.

(a) Find a matrix A to represent L with respect to the basis S. That is, find A such that $L(\boldsymbol{c}) = A\boldsymbol{c}$.

(b) Define $\boldsymbol{x} = (3, 2, 1)'$. Find the coordinate of $\boldsymbol{x}$ with respect to the basis S.

(c) Find the coordinate of $L(\boldsymbol{x})$ with respect to the basis S.