1. (8%)A box contains 5 red and 5 black marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.2; if they are different colors, then you win \$ − 1.00(that is, you lose \$1.00). Calculate the variance of the amount you win.

2. (8%)Suppose that $X_1, \cdots, X_n$ are independently, identically distributed random variables with probability mass function

$$f(x|\theta,p) = (1-p)p^{x-\theta}$$

where $x = \theta, \theta+1, \theta+2, \cdots$, and θ and p are unknown parameters. Find minimal sufficient statistics for (θ, p).

3. (10%)Let X_n have a chi-square distribution with n degrees of freedom. Show that the limiting distribution of $\sqrt{n}(\frac{X_n}{n} - 1)$ is $N(0,2)$.

4. (24%)Consider X and Y have a trinomial distribution with joint pmf

$$p(x,y) = \frac{n!}{x!y!(n-x-y)!}p_1^x p_2^y p_3^{n-x-y},$$

where x and y are nonnegative integers with $x+y \le n$; $p_1,\ p_2,\ p_3 \in (0,1)$ and $p_1+p_2+p_3 = 1$; and let $p(x,y) = 0$ elsewhere.

   (a) (6%)Find the moment generating function of a trinomial distribution.

   (b) (6%)Are X and Y independent?

   (c) (6%)Compute $E(X|y)$.

   (d) (6%)Calculate the correlation coefficient of X and Y.

5. (14%) Let $X_1, X_2 \cdots, X_n$ be iid with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \le x \le 1, 0 < \theta < \infty, \\ 0 & \text{elsewhere} \end{cases}$$

(a) (4%)Find the maximum likelihood estimator (MLE) of θ.

(b) (10%)Find the variance of MLE of θ.

6. (20%)Let $X_1, X_2 \cdots, X_n$ be iid from $U[0, \theta]$, $\theta > 0$.

(a) (6%)Find the uniformly minimum variance unbiased estimate (UMVUE) of θ.

(b) (7%)When $n = 5$, we reject $H_0 : \theta = 1$ and accept $H_a : \theta \neq 1$ if either $max(X_1, X_2, \cdots, X_5) \le \frac{1}{2}$ or $max(X_1, X_2, \cdots, X_5) > 1$. Find the power function of the test, $0 < \theta$.

(c)(7%) When $\theta = 1$, does $X_{(n)} = max(X_1, X_2, \cdots, X_n)$ converge to 1 in probability? Show the details.

7. (16%)X is said to have a Maxwell distribution if its density function is given by:

$$f(x|\theta) = (2/\pi)^{1/2}\theta^{3/2}x^2 exp(-x^2\theta/2), \ x > 0.$$

Let $X_1, X_2 \cdots, X_n$ be iid random samples from a Maxwell distribution with parameter θ. You can use the fact that $U = \sum_{i=1}^{n} X_i^2$ is a complete sufficient statistic for this family.

(a) (6%)Show that θU has a $\chi^2(3n)$ distribution.

(b) (10%)Show that the uniformly most powerful (UMP) test with size α for $H_0 : \theta = c$ vs $H_a : \theta > c$; rejects when $W < k$, where $W = cU$. What is k?