Abstract Algebra (II), Midterm Exam II

May 8, 2008

Instruction: *Show your reasoning. Answers without explanation may receive no credit. The starred problem is a bonus providing extra credit. Points total* $120 + 15$.

The symbol $\mathbb{F}_q$ *denotes a finite field of* q *elements.*

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1. (15 pts)

(a) State Sylow's Third Theorem (on the number of Sylow p-subgroups).

(b) Prove that a group of order 96 cannot be simple (i.e., has nontrivial proper normal subgroups).

2. (15 pts) Determine all ring homomorphisms from $\mathbb{Z}_{10}$ to itself. What about ring isomorphisms from $\mathbb{Z}_{10}$ onto itself?

3. (40 pts)

(a) If an ideal I of a ring R contains a unit, show that $I = R$.

(b) Let R be a commutative ring with unity. Show that R is a field if and only if the only ideals of R are 0 and R itself.

(c) Let R be a commutative ring with unity and M is a maximal ideal of R. Show that R/M is a field.

(d) Determine whether or not the following are maximal ideals:

$$\langle x\rangle \lhd \mathbb{Z}[x], \quad \langle x\rangle \lhd \mathbb{Q}[x], \quad \langle x^2+1\rangle \lhd \mathbb{Z}_2[x], \quad \langle x^2+1\rangle \lhd \mathbb{Q}[x].$$

4. (20 pts) Let F be a field of characteristic $p > 0$.

(a) Show that $K = \{x \in F \mid x^p = x\}$ is a subfield of F.

(b) Find K for $F = \mathbb{Z}_p, \mathbb{F}_{p^n}$, or $\mathbb{Z}_p(x)$.

5. (30 pts) Short-answer questions: Answer or prove or disprove the following questions. Explain briefly.

(a) How many elements of order 7 are there in a simple group of order 168?

(b) Give an example of a commutative ring R with unity and a maximal ideal I of R such that $I[x]$ is not a maximal ideal of $R[x]$.

(c) If R and S are commutative rings with unity, $\phi : R \to S$ is a ring homomorphism, and A is a maximal ideal of S, then $\phi^{-1}(A)$ is a maximal ideal of R.

(d) Which of the following rings are fields: (multiple choices)

$$\mathbb{Z}_3[i], \quad \mathbb{Z}_5[i], \quad \mathbb{Z}_7[i], \quad \mathbb{Z}_{17}[i]?$$

(e) The two rings $2\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic.

(f) The two fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{5}]$ are isomorphic.

(continued on the other side)

***6.** (15 pts)

(a) For any prime p, show that in $\mathbb{Z}_p[x]$,

$$x^{p-1} - 1 = (x-1)(x-2)\cdots(x-(p-1)).$$

(b) Deduce Wilson's Theorem: For every prime p,

$$(p-1)! \equiv -1 \pmod{p}.$$

(c) Prove that $(20!)^2 \equiv -1 \pmod{41}$.