There are 8 problems.

1. (15 pts) (a) Use the $\varepsilon - \delta$ definition to prove that $1/x$ is continuous at each $x \in (0,1)$.
(b) Prove that $1/x$ is not uniformly continuous on $(0,1)$.

2. (10 pts) Let $f(x, y)$ be a $C^2$ function on $\mathbb{R}^2$. Let $x = r \cos \theta$ and $y = r \sin \theta$. Prove that
   \[ f_{xx} + f_{yy} = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta \theta}. \]

3. (15 pts) (a) Suppose that $f$ and $f_n$, $n=1,2,3,...$, are defined on $(0, \infty)$, and are Riemann integrable on $[0, T]$. If $f_n \to f$ uniformly on $[0, T]$, prove that
   \[ \lim_{n \to \infty} \int_0^T f_n(x) \, dx = \int_0^T f(x) \, dx. \]
   (b) If in addition, $f_n \to f$ uniformly on $[0, T]$ for each $T > 0$ and there is a function $g$ defined on $(0, \infty)$, so that $|f_n| \leq g$, and
   \[ \int_0^\infty g(x) \, dx < \infty. \]
   Prove that
   \[ \lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty f(x) \, dx. \]
   Hint: \[ \int_0^\infty |f_n(x) - f(x)| \, dx = \int_0^T |f_n(x) - f(x)| \, dx + \int_T^\infty |f_n(x) - f(x)| \, dx. \]

4. (10 pts) A point $c$ is called a fixed point of a function $f$ if $f(c) = c$.
   (a) Prove that if $f$ is continuous on $[0,1]$ and $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, then $f$ has a fixed point.
   (b) Prove that if $f$ is differentiable on $[0,1]$ and $f'(x) < 1$ for $x \in [0,1]$, then $f$ has at most one fixed point in $[0,1]$.
   Hint: Consider the function $f(x) - x$.

5. (10 pts) Evaluate the surface integral
   \[ \int \int_S (x^2 + y^2 + z^2) \, d\sigma; \]
   where $S$ is the part of the plane $z = x + 2$ which lies inside the cylinder $x^2 + y^2 = 1$. 


6. (10 pts) (a) Find the power series expansion for the function \( f(x) = \ln(1 - x) \).
(b) Find the interval of convergence of the power series obtained in (a).

7. (10 pts) A rectangular box without a top is to have a volume of 12 cubic feet. Find the dimensions of the box that will have minimum surface area.

8. (20 pts) For any \( x = (x_1, x_2) \) in \( \mathbb{R}^2 \), we denote \( \|x\| = \sqrt{x_1^2 + x_2^2} \).
Let \( A \) be a compact subset in \( \mathbb{R}^2 \) and \( x_0 \notin A \).
(a) Prove that \( \text{dist}(x_0, A) = \inf\{\|x - y\| : y \in A\} > 0 \).
(b) Prove that there is a point \( y_0 \in A \) so that \( \|x_0 - y_0\| = \text{dist}(x_0, A) \).

Let \( B \) be a closed subset in \( \mathbb{R}^2 \) and \( x_0 \notin B \).
(c) Prove that \( \text{dist}(x_0, B) = \inf\{\|x - y\| : y \in B\} > 0 \).
(d) Prove that there is a point \( y_0 \in B \) so that \( \|x_0 - y_0\| = \text{dist}(x_0, B) \).

Let \( A \) be a compact subset in \( \mathbb{R}^2 \) and \( B \) be a closed subset in \( \mathbb{R}^2 \). Suppose that \( A \) and \( B \) are disjoint.
(e) Prove that \( \text{dist}(A, B) = \inf\{\|x - y\| : x \in A, \ y \in B\} > 0 \).
(f) Prove that there are points \( x_0 \in A \) and \( y_0 \in B \) so that \( \|x_0 - y_0\| = \text{dist}(A, B) \).

END