1. (10%) State the definition of variation $V(f; [a, b])$ of a real-valued function $f : [a, b] \to \mathbb{R}$. Consider the function $f(x) = x \sin(\frac{1}{x})$ for $0 < x \leq 1$ and $f(0) = 0$. Show that $f$ is continuous on $[0, 1]$, but $V(f; [0, 1]) = \infty$.

2. (15%) Give an example which shows that the image of a measurable set under a continuous transformation may not be measurable.

3. (15%) If $f \in L^2([0, 2\pi])$, show that $\sin(nx)f(x) \in L^2([0, 2\pi])$ for $n = 1, 2, \ldots$, and that $\int_0^{2\pi} \sin(nx)f(x)dx \to 0$ as $n \to \infty$.

4. (20%) Let $f(x, y), x, y \in [0, 1]$, satisfy the following conditions: for each $x$, $f(x, y)$ is an integrable function of $y$, and $\frac{\partial f}{\partial x}$ is a bounded function of $(x, y)$. Show that $\frac{\partial f}{\partial x}$ is a measurable function of $y$ for each $x$ and

$$\frac{d}{dx} \int_0^1 f(x, y)dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)dy.$$

5. (20%) Let $f_n(x)$ be a sequence of functions in $L^2([0, 1])$, which converge almost everywhere to a function $f(x)$ in $L^2([0, 1])$, and suppose that there is a constant $M$ such that $\|f_n\|_2 \leq M$ for all $n$. Then for each function $g$ in $L^2([0, 1])$ we have

$$\int_0^1 f(x)g(x)dx = \lim_{n \to \infty} \int_0^1 f_n(x)g(x)dx.$$

6. (20%) For $1 \leq p < \infty$, define

$$\omega(f, p) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f|^p\right)^{1/p}.$$

Show that if $p_1 < p_2$, then $\omega(f, p_1) \leq \omega(f, p_2)$. 