(30%) 1. Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Must $f$ be Riemann integrable? Let $g(x) = \int_0^x f(t)dt, \forall x \in [0, 1]$. Must $g$ be absolutely continuous on $[0, 1]$? Prove or disprove your answers.

(40%) 2. Let $f, g \in L^\infty([0, 1])$ satisfy the following properties:

(P1) $f(x) \leq f(y)$ for $x, y \in [0, 1] \setminus \omega$ and $x \leq y$, where the set $\omega$ has only finite points;

(P2) $g(x) \leq g(y)$ for $x, y \in [0, 1] \setminus \gamma$ and $x \leq y$, where the set $\gamma$ has countably infinite many points. Answer yes or no and prove your answers for the following questions:

(a) Can (P1) be equivalent to (P2)?
(b) Must $f$ and $g$ be (Lebesgue) integrable?
(c) Must $f$ be differentiable almost everywhere?
(d) Must $g$ be differentiable almost everywhere?
(e) Suppose the set $\gamma$ has only one accumulation point. Must $g$ be differentiable almost everywhere?

(8 points for each)

(30%) 3. Let $f \in L^2(\mathbb{R}^2)$ be such that $f \neq 0$ in $\mathbb{R}^2$ and

$$
\int_{\partial B_r(x_0)} f^2 ds \leq \int_{\partial B_r(x_0)} f^2 ds \text{ for } 0 < r < 1, x_0 \in \mathbb{R}^2,
$$

where $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$ and $\partial B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| = r\}$. Assume the zero set $N = \{x \in \mathbb{R}^2 : f(x) = 0\}$ is nonempty. Can the set $N$ have interior points? Can $\int_{B_r(x_1)} f^2 = O(r^k)$ for all $x_1 \in N$ and $k \in \mathbb{N}$? Prove or disprove your answers.