Real Analysis

This exam contains 5 problems with a total of 100 points. Each problem costs 20 points. Do all problems and show all your work for partial credits.

1. Let \( f \) be a nonnegative function which is integrable over a compact set \( E \subset \mathbb{R}^n \). Show that given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for every measurable subset \( A \subset E \) with measure \( mA < \delta \) we have \( \int_A f < \varepsilon \).

2. State the Lebesgue Dominated Convergence Theorem and give an example to illustrate how to apply it.

3. Let \( f_n \) be a sequence of measurable function in \( L^\infty(E) \). Prove that \( f_n \) converges to \( f \) in \( L^\infty(E) \) if and only if there is a measure zero subset \( A \subset E \) such that \( f_n \) converges to \( f \) uniformly on \( E - A \).

4. Let \( f_n \) be a sequence of measurable function in \( L^2([0,1]) \), which converges almost everywhere to a function \( f \) in \( L^2([0,1]) \). Show that \( f_n \) converges to \( f \) in \( L^2([0,1]) \) if and only if \( \|f_n\| \to \|f\| \).

5. Prove that \( L^\infty([0,1]) \) is complete.