1. Let \( f : \mathbb{R} \to [0, \infty] \) be a measurable function and \( 0 < \alpha < \infty \) be a constant. Suppose that \( \int_{\mathbb{R}} f \, dx = c, \ 0 < c < \infty \). Find
\[
\lim_{n \to \infty} \int_{\mathbb{R}} n \log[1 + (f/n)^\alpha] \, dx.
\] (20 points)

2. Suppose that \( f \) is a measurable function on \( \mathbb{R} \) and
\[
\varphi(p) = \int_{\mathbb{R}} |f|^p \, dx, \quad (0 < p < \infty).
\]
Let \( E = \{ p : \varphi(p) < \infty \} \). Assume \( \|f\|_\infty > 0 \). Prove that \( \log \varphi \) is convex in the interior of \( E \). (20 points)

3. Let \( f \in L^2(0, 2\pi) \). Is it possible to have simultaneously
\[
\int_0^\pi (f(x) - \sin x)^2 \, dx \leq 4/9
\]
and
\[
\int_0^\pi (f(x) - \cos x)^2 \, dx \leq 1/9.
\] (20 points)

4. Let \( \phi \) be a convex function on \( (-\infty, \infty) \) and \( f \) be an integrable function on \([0, 1]\). Prove that
\[
\phi[\int_0^1 f(x) \, dx] \leq \int_0^1 \phi(f(x)) \, dx.
\] (20 points)

5. Let \( E \) be a measurable set of finite measure and \( \{f_n\} \) be a sequence of measurable functions defined on \( E \). Let \( f \) be a real-valued function such that for each \( x \) in \( E \) we have \( f_n(x) \to f(x) \). Prove that given \( \epsilon > 0 \) and \( \delta > 0 \), there is a measurable set \( A \subset E \) with \( mA < \delta \) and an integer \( N \) such that for all \( x \in E \setminus A \) and all \( n \geq N \),
\[
|f_n(x) - f(x)| \leq \epsilon.
\] (20 points)