1. Let $A_n \subset \mathbb{R}^2$, $A_n$ the interior of the circle with radius 2 and center at

$$(0, 1 + (-1)^n/n).$$

Find $\limsup_n A_n$ and $\liminf_n A_n$. (20 points)

2. Call two real numbers $x$ and $y$ equivalent iff $x - y$ is rational. Choose a number of each distinct equivalence class $B_x = \{y : y - x \in \mathbb{Q}\}$ to form a set $A$; assume that the representatives are chosen so that $A \subset [0, 1]$. Establish the following:

(a) If $r$ and $s$ are distinct rational numbers, $(r + A) \cap (s + A) = \emptyset$; also

$$\mathbb{R} = \bigcup \{r + A : r \in \mathbb{Q}\}. \quad (10 \text{ points})$$

(b) If $A$ is Lebesgue measurable, then the measure $\mu(r + A) = 0$ for $r \in \mathbb{Q}$.

Conclude that $A$ can not be Lebesgue measurable. (10 points)

3. If $f$ is nonnegative and has an improper Riemann integral, show that $f$ is Lebesgue integrable and the two integrals are equal. (10 points)

Give a counterexample to this result if the nonnegativity hypothesis is dropped. (10 points)

4. Give an example of a sequence of functions $f_n$ on $[a, b]$ such that each $f_n$ is Riemann integral, $|f_n| \leq 1$ for all $n$, $f_n \rightarrow f$ everywhere, but $f$ is not Riemann integral. (20 points)

5. If $|f_n| \leq g$ for all $n = 1, 2, \cdots$, where $g$ is integrable and $\{f_n\}$ converges to $f$ in measure, show that $\int f_n(x)dx \rightarrow \int f(x)dx$. (20 points)