Real analysis

1. Find the limit
\[
\lim_{n \to +\infty} \int_0^{\frac{\pi}{2}} \sin^n x \, dx.
\]

2. (20 pt)
(i) For what positive number \(n\), the function \(f(x) = x^n\) is uniformly continuous on \([2009, +\infty)\)? (justify your answer!)

(ii) For what positive number \(n\), the function \(f(x) = x^n\) is uniformly continuous on \((0, 2009)\)? (justify your answer!)

3. (20 pt) Suppose that \(f(x, y)\) is a bounded real-valued function defined on \(E \times (0, 1]\) where \(E\) is a measurable subset of \(\mathbb{R}\). Assume that for each fixed \(x \in E\), \(f(x, y)\) is a continuous function of \(y\); and for each fixed \(y \in (0, 1]\), \(f(x, y)\) is a measurable function of \(x\). Prove that
\[
F(x) := \lim_{y \to 0^+} f(x, y)
\]
is measurable on \(E\).

4. (20 pt) Suppose that \(f \in L^1(\mathbb{R})\) and \(f_n \in L^1(\mathbb{R})\) for \(n \in \mathbb{N}\). Assume that for each \(n \in \mathbb{N}\), we have
\[
\int_{\mathbb{R}} |f_n(x) - f(x)| \, dx \leq \frac{1}{n^2}.
\]
Show that \(f_n(x) \to f(x)\) a.e. \(x \in \mathbb{R}\).

5. (20 pt) Let \(f \in L^1(\mathbb{R})\) and suppose that
\[
\int_A f(x) \, dx = 0
\]
for all bounded measurable subsets \(A\) of \(\mathbb{R}\). Show that \(f\) is equal to 0 almost everywhere.